Some Impartial Combinatorial Games for 2 Players

Ten-coin game
Setup: There is a pile of ten coins.
Rules: On your turn, you may remove 1 or 2 coins from the pile.
Winning: The person to take the last coin wins.

Nim
Setup: There are several piles containing several coins each. (Typically three piles containing 3, 5 and 7 coins).
Rules: On your turn, you may remove any number of coins from any one pile.
Winning: The person to take the last coin wins.

Circular Kayles
Setup: There is a circle of N coins.
Rules: On your turn, you may remove either any one coin, or any two coins that were originally adjacent in the circle.
Winning: The person to take the last coin wins.

Northcott's Game (Really a partial game, but can be analyzed as impartial)
Setup: There is a chessboard with one black and one white checker in each row. One player owns all white checkers, the other owns all black ones.
Rules: On your turn, you must move one of your pieces to another vacant spot in the same row, except that you may not jump over your opponent's pieces.
Winning: The person to make the last move wins.

Wythoff's Game
Setup: There are two piles, respectively containing M and N coins.
Rules: On your turn, you may remove any number of coins from any one pile, or an equal number of coins from both piles.
Winning: The person to take the last coin wins.
(Can you determine a formula for the winning positions in this game?)

Treblecross (1D Tic-Tac-Toe)
Setup: There is a strip of paper with N empty boxes on it.
Rules: On your turn, you mark an X in any box.
Winning: The first person to complete a sequence of three adjacent marked boxes wins.

Grundy's Game
Setup: There is a pile containing N coins.
Rules: On your turn, you may take any pile and break it into two nonempty, unequal-sized piles.
Winning: The last person to make a move wins.
(Open conjecture: The Nim values for this game are ultimately periodic)

Wyt Queen
Setup: Given is a chessboard with one white queen on it.
Rules: On your turn, you may move the queen any positive number of cells left, up, or diagonally up and left.
Winning: The last person to make a move wins.
(This is equivalent to one of the other games above…)

What is a combinatorial game?

1) There are *two players* (often called left and right) who play *alternately*
2) There are several, but finitely many, game states, we will call *positions*
3) There is complete information (both players know all *options* from each position)
4) Play is *deterministic*, there are no chance moves
5) The game must terminate in a *finite* number of moves
6) The last person to make a move wins

Rules for analyzing combinatorial games using W-L state analysis:

1) A game-winning move goes to a W position.
2) A game-losing move goes to an L position.
3) If a position has *any* options marked W, mark it L
4) If a position has *only* options marked L, mark it W

You can be guaranteed a win if you are ever able to bring the game to a W position on your turn.
Addendum for draws:

5) A game-drawing move goes to a D position
6) If a position does not satisfy rules 3) or 4) (goes to one or more D states and maybe some L states), mark it D

Impartial Games

If both players in a game are always allowed to make the same moves from any position, we call it 'impartial'.

The Tweedledum-Tweedledee Strategy for Impartial Games

Suppose that an impartial combinatorial game can be broken down into two parallel, identical subgames. Then the second player can always win, by mirroring whatever move his opponent makes, in the opposite subgame. In other words the game is an empty Nim pile with added reversible moves.

The Sprague-Grundy Theorem for Impartial Games (Sprague 1936, Grundy 1939)

> All positions in an impartial combinatorial game are equivalent (up to reversible moves) to a single pile of Nim counters.

We call the number of counters in this equivalent pile that position's *Nim value*. When playing this game, you can be guaranteed a win if you are ever able to bring the game to a position with value 0.

Calculating Nim Values – Parallel Games

Suppose you are given two games to be played in parallel – on your turn you may move in either one of the two games, and the last person to make a move in either game wins. If the Nim values of the initial positions for the two games are A and B, then the value of the combined game is A xor B (write both values in binary, add without carrying, and convert back from binary to get the resulting value). For this reason the bitwise-xor operation is also sometimes called the nim-sum.

Calculating Nim Values – The Mex Rule (mex = *m*inimum *ex*cluded)

Suppose that we want to calculate the Nim value for a particular position P in an impartial combinatorial game, and that the Nim values of all of its options are known to be $V_1, V_2, V_3, \ldots V_N$.
Then the Nim value of P is

$$\text{mex}(V_1, V_2, V_3, \ldots V_N) == \text{the least non-negative integer not equal to } V_i \text{ for any } i.$$

Reference: Winning Ways for your Mathematical Plays. Berkelamp, Conway & Guy. 1982, 2001.