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Introduction to Combinatorial Games

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To the Student

This class will probably not be like any math course you've had before. It's not algebra. It's not calculus. There are numbers involved, but they don't always behave like the numbers you've known since you were little. It will challenge your ability to think logically, to reason carefully, and to present thorough arguments. And it will be FUN!

You will need to work at this. Pay key attention to the definitions and what you can logically deduce from them. Don't be afraid to play around with the ideas (often literally) and see where you can get. Try some examples. Develop some patterns. For many of these problems, even a partial contribution will be a positive contribution.

There are also some very abstract problems here. By the time you get there, you'll be thinking very differently about the whole idea of games and strategy.

Best of luck. Let the games begin!

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Chapter 1

Nim

1.1 *Take Two*

Our first game. The first game in this course will be called Take Two. Here's how one plays Take Two.

> **The game Take Two**
>
> *Setup.* There are several piles of beans[1] on a table. Here and throughout this course, "several" means "at least one but not infinitely many". Each pile[2] has several beans in it. This game is played with two players. Both players know how many beans are in each pile; the piles are not hidden.
>
> *Play.* One player starts. On a player's turn, he[3] must take exactly one or two beans from a single pile. Play then passes to the other player.
>
> *Conclusion.* The first player who is unable to make a move is the loser[4]. In this game, this occurs when all of the beans have been taken.

Problem 1. *Suppose there is just one pile of beans, and it contains 7 beans. Play the game with a partner several times. Who wins most often?*

Problem 2. *Describe a strategy[5] used when playing Take Two with a pile of 7 beans.*

[1]Some authors prefer chips instead of beans. I like beans on chips.

[2]Often, the word *heap* is used instead of *pile*.

[3]Historically, the English language did not have a gender neutral third person singular pronoun. In the past, people would write *he/she*, which is tedious. More recently, *they* has become acceptable as a singular pronoun. However, this can cause confusion about whether it is referring to one person or more than one. Rather than go with the times, we err on the side of less confusion and in cases such as this when either player may be referred to, we will simply pick a gender, rather than using *he/she* or *they*.

[4]This will be the rule for nearly every game in this course.

[5]Right now, feel free to describe your strategy however you wish and you can interpret the word "strategy" however you want. In the next definition, the word "strategy" will be clarified to mean something specific.

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Before going too much further[6], we better make a definition or two.

Definition 3. *If a player, say Player A, has a* winning strategy*, she has moves that she can make so that she will win the game no matter what her opponent does. The moves that she makes may depend on the moves her opponent makes.*

There are three conceivable types of Take Two games. It may be possible that the player who moves first has a winning strategy. It may be possible that the player who moves second has a winning strategy. Also, it may be possible that neither player has a winning strategy.

Problem 4. *Does either player[7] have a winning strategy for Take Two with a single pile of 7 beans? Describe it.*

Problem 5. *Does either player have a winning strategy for Take Two with a single pile of 6 beans?*

Problem 6. *Discuss the winning strategy for Take Two with a single pile of n beans, where n is a positive integer.*

Problem 7. *Now suppose there are two piles, each of which has 8 beans. Discuss the winning strategy.*

In class, we discovered the following theorem.

Theorem 8. *In the game of Take Two, if there are two piles with an equal number of beans, then the second player to play has a winning strategy.*

Problem 9. *Describe how the winning strategy works in Theorem 8.*

Problem 10. *Now suppose there are two piles, one of which has 9 beans and the other of which has 8 beans. Discuss the winning strategy.*

Problem 11. *Suppose there are two piles of 5 beans and one pile of 3 beans. Who has the winning strategy?*

Problem 12. *Discuss the winning strategy for two piles of n_1 and n_2 beans.*

Problem 13. *In the previous Problems, I've always asked you to discuss the winning strategy. Is it always the case that one player has a winning strategy? In other words, is it possible that neither player has a winning strategy? Justify your claim as best you can.*

Problem 14. *Discuss the winning strategy for three piles of n_1, n_2, and n_3 beans.*

[6]It seems like there will be quite a few footnotes in this document.

[7]In this case, "either player" refers to either the first player to move or the second player to move.

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Problem 15. *Discuss the winning strategy for k piles of* $n_1, n_2, \cdots, n_k$ *beans.*

Definition 16. *A position for an impartial game can be in one of two* outcome classes*: either the next player to move has a winning strategy (in which case we call the position an* $\mathscr{N}-$*position), or the player who just moved (the previous player to move) has a winning strategy (in which case we call the position a* $\mathscr{P}-$*position).*

Problem 17. *Explain why each impartial game must be in one of the two outcome classes. In other words, explain why exactly one of the players has a winning strategy.*

1.2 Take Three

It only seems natural that our next game should be Take Three. Here's how to play.

The game Take Three

Setup. There are several piles of beans. Each pile has several beans in it. This game is played with two players. Both players know how many beans are in each pile; the piles are not hidden.

Play. One player starts. On a player's turn, she must take exactly *one or two or three* beans from a single pile. Play then passes to the other player.

Conclusion. The first player who is unable to make a move is the loser. In this game, this occurs when all of the beans have been taken.

Problem 18. *Suppose there is just one pile of beans, and it contains 7 beans. Play the game with a partner several times. Who wins most often?*

Problem 19. *Describe a strategy used when playing Take Three with a pile of 7 beans.*

Problem 20. *Discuss the winning strategy for Take Three with a single pile of n beans.*

Problem 21. *Discuss the winning strategy for Take Three with two piles, each of which has 11 beans.*

Problem 22. *Discuss the winning strategy for Take Three with two piles of* n_1 *and* n_2 *beans.*

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1.3 Jordan

A similar game. The next game didn't have a name, so the author of these notes named it after his dog, Jordan. Here's how to play.

> **The game Jordan**
>
> *Setup.* There are several piles of beans. Each pile has several beans in it.
>
> *Play.* One player starts. On a player's turn, she must take exactly two or three beans from a single pile. Play then passes to the other player.
>
> *Conclusion.* The first player who is unable to make a move is the loser. In this game, this occurs when all of the piles contain at most one bean.

Problem 23. *Discuss the winning strategy for Jordan with one pile of n beans.*

Problem 24. *Discuss the winning strategy for Jordan with two piles of n beans each.*

Problem 25. *What other game configurations can you figure out the strategy for? Two piles of 7 and 8 beans? Three piles of 1, 2, and 3 beans? Whenever all piles are equal? Make a conjecture and see if you can justify it!*

Problem 26. *Describe the general strategy for playing Jordan with any number of piles of any number of beans.*[8]

1.4 A brief foray into octal games

Suppose we modify the rules of Take Two, Take Three, and Jordan in a natural way.

[8] It should be noted that some of these Problems are quite difficult. You should spend some time thinking about the problems, but you are not expected to get every one of them right away. If needed, we will come back to difficult problems to readdress them later after we've learned more.

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Extension of Take Two, Take Three, and Jordan

Setup. There are several piles of beans. Each pile has several beans in it.

Play. One player starts. On a player's turn, she must take exactly a or b beans from a single pile. Play then passes to the other player.

Conclusion. The first player who is unable to make a move is the loser.

Problem 27. *Which values of a and b are easy to analyze for one pile of beans? Why?*

Problem 28. *Which values of a and b are easy to analyze for two piles of beans? Why?*

Problem 29. *Describe at least two pairs of numbers a and b for which the analysis of the game is too hard to do at this time. We'll address these later.*

1.5 Nim

The game of *nim* is played as follows.

The game of nim

Setup. There are several piles of beans. Each pile has several beans in it.

Play. One player starts. On a player's turn, he may take any number of beans from a single pile. Play then passes to the other player.

Conclusion. The first player who is unable to make a move is the loser. In nim, this occurs when all of the beans have been taken.

Problem 30. *Discuss the winning strategy for a game of nim with one pile of n beans*[9].

Problem 31. *Which player has a winning strategy in a game of nim with two piles of 5 beans each? Explain.*

[9]This is the easiest question in this course.

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Problem 32. *Use your answer from Problem 31 to give the winning strategy in as many nim games as you can. For instance, you might be able to describe the winning strategy in games with 4 piles, each with 5 beans.*

Problem 33. *Who has the winning strategy for two piles of 2 and 3 beans?*

Problem 34. *Who has the winning strategy for two piles of 2 and 4 beans?*

Problem 35. *Who has the winning strategy for two piles of 3 and 4 beans?*

Problem 36. *Who has the winning strategy for three piles of 1, 2, and 3 beans?*

Problem 37. *Who has the winning strategy for three piles of 2, 4, and 6 beans?*

Theorem 38. *In a game of nim with two piles of unequal number of beans, the first player to move has a winning strategy.*

We're now in a position to build some rich mathematical theory pertaining to nim.

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Chapter 2

Nimbers

2.1 Nimbers for single pile nim games

We can easily assign a number value to each single pile nim game we encounter according to the number of beans in the pile. We'll do calculations with and in doing so these numbers are going to have somewhat strange properties, so we'll refer to them as *nimbers* rather than numbers.

Definition 39. *A nim game that consists of a single pile of n beans is said to have* nimber *value* $*n$.

Problem 40. *Which nimber values represent single pile nim games which are* $\mathscr{N}-$*positions (games in which the first player to move in the game has a winning strategy) and which represent* $\mathscr{P}-$*positions (games in which the second player has a winning strategy?)*

2.2 Adding games

In order to extend this game-is-represented-by-a-nimber theory, we must next talk about adding games.

Definition 41. *A game* position *is a configuration of the objects of the game. The positions for the games we've studied so far are described in the* setup *items.*

For example, a nim position consists of a number of piles of beans. One nim position might be described as two piles of 3 and 4 beans.

Definition 42. *A* game *is a position together with a description of play together with an end condition*[1]*. These are described by the* setup*,* play*, and* conclusion *items in the boxes.*

[1] Later, once we generalize this theory, we'll put a few more restrictions on what we mean when we talk about the word *game*.

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Games will be represented by capital letters, usually G and H. For instance, G might refer to the game of nim with three piles, each of 17 beans. Notice that neither the definition of *game* nor the definition of *position* capture whose turn it is. Now that we know what games are and how their notation looks, let's learn how to add games.

Definition 43. *The* sum *of two games is a game created by setting the two games next to each other on the table. On one's turn, a player may move in either of the two games, but not both.*

Before we explore how adding games works with nimbers, let's make sure we understand the definition.

Problem 44. *Let G be the game of nim with a single pile of 4 beans. Let H be the game of nim with a single pile of 2 beans. Who has a winning strategy in the game $G+H$?*

Problem 45. *Let G be the game of nim with a single pile of 5 beans and let H be the game of Take Two with a single pile of 3 beans. Suppose it's your turn to move. Describe the moves that you have available to you in the game $G+H$.*

Problem 46. *Suppose P is the game of Take Two with a single pile of 12 beans and Q is a game of Take Three with two piles, each of 17 beans. What is the outcome class of the game $P+Q$? Describe the strategy employed by the player with the winning strategy.*

Problem 47. *Describe how a game of Jordan with two piles, each of 10 beans can be described as the sum of two games.*

Problem 48. *Recall the definition of* nimber *for a single pile nim game*[2]. *Consider two games G and H which are both nim games with a single pile of 4 beans. Both of these games have nimber value $*4$. Who has the winning strategy in $G+H$?*

Problem 49. *What is the only nimber which is assigned to a single pile nim game in which the second player has a winning strategy?*

By the reasoning of the last two Problems, it seems somewhat natural that since $G+H$ is a second player win nim game, it should have nimber value $*0$. When we add games together which already have defined nimber values, it would be interesting to add their corresponding nimbers together to see what we get. Unfortunately, nimber addition doesn't work like regular addition[3]. We can use the fact that if $G+H$ is a $\mathscr{P}-$position (a game in which the second player to move has a winning strategy) then it should have nimber $*0$ to make a definition of addition of nimbers.

[2]This means, go look it up if you don't remember it.

[3]Or maybe this is fortunate, since it makes this quite interesting.

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Definition 50. *If G has nimber value $*m$ and H has nimber value $*n$ and $G+H$ is a second player win game, then we say the* sum *of the nimbers $*m \oplus *n = *0$. In this case, we could say $*m = - * n$.*

Notice that we have just defined what it means to add two nimbers in the case when they add together to make $*0$. Also notice that when we add them, we use the $\oplus$ symbol rather than the $+$ symbol to avoid confusion with regular addition..

Problem 51. *If G is nim with a single pile of 17 beans, what is the nimber value of $G+G$?*

Problem 52. *Argue why $*n = - * n$ for every[4] positive integer n.*

Problem 53. *Suppose G is a nim game and H is a nim game with nimber value $*4$. If it can be shown that $G+H$ has nimber value $*0$, what is the nimber value of G?*

Problem 54. *If G is nim with a pile of one bean and a pile of two beans, what is the nimber value of G? Hint: see Problem 36.*

Problem 55. *If G is nim with a pile of one bean and a pile of three beans, what is the nimber value of G?*

Problem 56. *What is $*2 \oplus *3$?*

Problem 57. *Argue why $\oplus$ is associative[5].*

Problem 58. *Find the values of $*2 \oplus *3$ (already done), $*2 \oplus *2$, $*2 \oplus *1$, $*2 \oplus *0$, $*1 \oplus *4$, and $*0 \oplus *4$.*

Problem 59. *Use the previous Problem to describe which move you should make if it's your turn in a nim game with two bean piles of 2 and 4 beans.*

Problem 60. *If you have two piles of beans in a nim game, is it always possible to move to a game with value $*0$? Prove it or explain why not.*

Problem 61. *Is there an easy answer to the previous Problem if there are three piles?*

2.3 Nim addition in general

Problem 62. *Find the nim sums of a lot of nimbers. Do you see any pattern? For instance, we've seen $*1 \oplus *2 = *3$ and $*2 \oplus *2 = *0$. Can you develop a general formula for computing $*n \oplus *m$?*

[4]Nimbers aren't like numbers, are they?!

[5]If you don't remember what associativity is, now would be a good time to look it up somewhere.

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Before trying to figure out the general formula for nim addition, let's watch Lefty and Rita play nim. We start with four piles containing 13, 9, 6, and 1 beans. Lefty gets to move first.

Lefty first arranges the piles as shown below, where ♠ represents one bean.

pile 1	pile 2	pile 3	pile 4
♠ ♠ ♠ ♠ ♠ ♠ ♠ ♠ ♠ ♠ ♠ ♠	♠ ♠ ♠ ♠ ♠ ♠ ♠ ♠ ♠	♠ ♠ ♠ ♠ ♠ ♠	♠

Now, Lefty decides to take one bean from the pile of 6, leaving piles of 13, 9, 5, and 1 beans. Below is how Lefty has arranged those piles in his head.

pile 1	pile 2	pile 3	pile 4
♠ ♠ ♠ ♠ ♠ ♠ ♠ ♠ ♠ ♠ ♠ ♠	♠ ♠ ♠ ♠ ♠ ♠ ♠ ♠ ♠	♠ ♠ ♠ ♠ ♠	♠

Rita's response is to take 6 beans off of the first pile. After she does that, Lefty sees piles of 7, 9, 5, and 1 beans, which he organizes in his head as shown below.

pile 1	pile 2	pile 3	pile 4
♠ ♠ ♠ ♠ ♠ ♠ ♠	♠ ♠ ♠ ♠ ♠ ♠ ♠ ♠ ♠	♠ ♠ ♠ ♠ ♠	♠

Lefty takes 6 beans from the pile of 9, giving piles of size 7, 3, 5, and 1. Lefty sees this as:

pile 1	pile 2	pile 3	pile 4
♠ ♠ ♠ ♠ ♠ ♠ ♠	♠ ♠ ♠	♠ ♠ ♠ ♠ ♠	♠

Rita responds by taking all 5 of the beans from the pile of 5, leaving piles of size 7, 3, and 1.

Problem 63. *Who is going to win this game? Why?*

Problem 64. *Explain the strategy for playing nim.*

Now that we have a strategy for playing nim, we should make sure that our scheme for adding nimbers behaves accordingly. In particular, we need to make sure that every $\mathscr{P}-$position has nimber value $*0$ and every $\mathscr{N}-$position has some other nimber value. And so far we only are able to add nimbers using these facts; it would be nice to have a better formula to add nimbers that is based on the way we play nim itself.

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The next Lemma is true, and we we'll hear a brief argument as to *why* it is true in class. You can use it in your explanations of the subsequent problems.

Lemma 65. *There is exactly one way to express a whole number n as a sum of distinct, nonnegative powers of two.*

Problem 66. *Explain what the statement of Lemma 65 means. Give an example to illustrate your understanding.*

Theorem 67. *To add* $*n \oplus *m$*, first express both n and m as a sum of powers of 2. To obtain the nim sum, add together (in the usual way) the powers of two of both nimbers, except if a power of 2 occurs in both sums, ignore both of them.*

Problem 68. *Give an example to illustrate how to calculate the sum of two nimbers using the theorem above.*

Problem 69. *Complete a nimber addition table for nimbers* $*1$ *through* $*9$.

$\oplus$	$*1$	$*2$	$*3$	$*4$	$*5$	$*6$	$*7$	$*8$	$*9$
$*1$									
$*2$									
$*3$									
$*4$									
$*5$									
$*6$									
$*7$									
$*8$									
$*9$									

Problem 70. *Find* $*1 \oplus *2 \oplus *3$. *Which problems earlier show that your answer is correct?*

Problem 71. *Find* $*17 \oplus *32 \oplus *12$ *and* $*3 \oplus *8 \oplus *9 \oplus *11$.

2.4 Nim strategy

Nim addition and nim strategy go hand in hand.

Problem 72. *In the nim game with three piles of beans with sizes 17, 32, and 12, which player has a winning strategy? Explain the first move of the player with the winning strategy*[6].

Problem 73. *Same as the last question, but now with piles of 3, 8, 9, and 11.*

[6] If the first player has a winning strategy, say what the first move is for this player. If the second player has a winning strategy, explain what her first move is in response to whatever the first player does is.

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Problem 74. *Do one more example like the last two, but with your own numbers. Try to impress me*[7].

Problem 75. *Write, as eloquently as possible, a description of how one should play nim.*

[7] You might find that I am easily impressed. Or you might not.

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Chapter 3

Sprague-Grundy Theorem

3.1 Definitions

We've been dealing with nim an awful lot. There's a good reason why, and this was discovered independently by Roland Sprague in 1935[1] and Patrick Grundy in 1939[2][3]. In this section, we build to their famous theorem.

Definition 76. *A game has* perfect information *if both players have full information about the legal moves at any given time and at any time in the future.*

For example, chess has perfect information because both players can see the game board and know what the legal moves are. Tic-tac-toe and nim are also perfect information games. So are Take Two and Jordan. Games such as poker and backgammon are not perfect information games. In poker some of the information is hidden to one or more of the players because each player cannot see the other players' cards. In backgammon some of the information is hidden from the players because the outcomes of the future dice rolls is unknown to the players when they are making their moves. Generalizing these thoughts, any game with hidden information (such as hidden cards) or a random element (such as rolling dice) is not a perfect information game.

Definition 77. *A game is said to satisfy the* ending condition *if there is no infinitely long string of legal moves*[4]*. All such games must end in finite time.*

[1] Sprague published the paper *Über mathematische Kampfspiele* in the Tôhoku Mathematical Journal, volume 41, pages 438 – 444.

[2] Grundy published *Mathematics and Games* in the journal Eureka, volume 2, pages 6–8.

[3] Though Grundy published after Sprague, since Sprague's paper appeared in German in an obscure Japanese journal, it is believed that Grundy had no knowledge of Sprague's work and therefore should be given just as much credit for this work. Indeed, we usually refer to the theorem of this section as either the Sprague-Grundy theorem or just the Grundy theorem.

[4] Technically, we don't even require that this infinitely long string of legal moves alternate from one player to the next. If it is possible for one player to make infinitely many legal moves in succession, the game does not satisfy the ending condition.

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All of the games in this document so far satisfy the ending condition, since we start with only a finite number of beans and at least one bean is removed on each turn. Monopoly is an example of a game which does not satisfy the ending condition[5]. Chess also does not satisfy the ending condition: it is possible that a chess game goes on infinitely long.

Definition 78. *A game is said to be played under* normal play *if the first player who is unable to move is the loser*[6].

Definition 79. *A game is said to be* impartial *if the set of legal moves does not depend on which player is to move.*

All of the games we've dealt with so far are impartial, since nowhere did we mention that one player is allowed a certain set of move and the other player gets a different set. Indeed, the only distinction between players is between the first and second player to move. Chess is not impartial, because the player playing the white pieces is not allowed to move the black pieces and vice versa. Tic tac toe is not impartial, either. Even though both players are allowed to make moves in the same kinds of squares (the ones which have no marks in them already), one player is allowed to draw only X and the other player is allowed to draw only O. The game of dots and boxes is impartial. So is hopscotch: players toss a rock and skip through the boxes while picking up the rock. Each player has the same set of moves (tossing the rock). Hopscotch is not a perfect information game, due to the random element introduced by tossing the rock. Dots and boxes is a perfect information game, but it is not played under normal play because the loser is determined by who has made the fewest boxes, rather than by who is the player unable to move.

Earlier we had a working definition of the word *game* that was consistent with how we've been using it. Here, we'll give an alternate definition which will be useful for dealing with some of the theory of games. It's a bit of a strange way of thinking of a game, and a bit of a circular definition.

Definition 80. *A* game *G can be thought of as a set* $\{G_1, G_2, \cdots, G_k\}$ *of games which form the* options *for G. Thus G is the set of all games which can be arrived at from G through one move.*

Problem 81. *For the game of nim with two piles of 3 beans, describe the set of options.*

Problem 82. *Describe the set of options for the game of nim with a single pile of n beans, where n is a positive integer.*

[5] Though if you add in the stipulation that the game is done when your brother says, "This is dumb. Let's not play this anymore." — just because he is losing —- then I guess it would satisfy the ending condition, since this always occurs in finite time.

[6] The opposite of normal play is *misère play*, when the first player unable to move is the winner. The theory of misère play games is much more difficult than that of normal play games.

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Problem 83. *Is the nim game consisting of piles of 3, 4, 17, 45, and 97 beans a $\mathscr{P}-$position or an $\mathscr{N}-$position? Explain.*

We now introduce the Grundy Function of a game. It has a curious definition.

Definition 84. *The Grundy Function $\mathscr{G}$ is a function from the set of impartial games to the set of nimbers. Suppose G is a game with option set $\{G_1, G_2, \ldots, G_k\}$. Define $\mathscr{G}(G) = \min\{n \geq 0 : n \neq \mathscr{G}(G_i)$ for any $i\}$.*

We usually write $\mathscr{G}(G) = \text{mex}\{G_i : i = 1, \cdots, k\}$, where *mex* stands for *minimal excluded value*. The value of the Grundy function for a particular game is often called its *Grundy value*.

Problem 85. *What is* $\text{mex}(G)$ *for the game G which is nim consisting of a single pile of n beans?*

3.2 Turning Turtles

Before moving forward, let's get some practice using the mex function on a game other than nim. Here's a game about turning turtles on their backs.

The game of Turning Turtles

Setup. There are several turtles lined up in a row. Each turtle is either on its feet or on its back.

Play. On one's turn, a player must turn one turtle from its feet to its back. Then, if she desires, the player may also select another turtle to the left of this spot and flip it over. This optional second turtle can be flipped from its feet to its back or from its back to its feet. Play then passes to the other player.

Conclusion. The first player who is unable to make a move is the loser. In Turning Turtles, this occurs when all of the turtles are upside down.

Problem 86. *Calculate the Grundy values for all games in which the rightmost upright turtle occupies the first, second, or third position.*

We'll see in the next section that if a game has a Grundy value of 0 then it is a $\mathscr{P}-$position and if it has a nonzero Grundy value it is an $\mathscr{N}-$position. Use this fact now (without justification) to do the next two problems.

Problem 87. *Use the Grundy values in the games from the previous Problem to determine which of those games are $\mathscr{P}-$position games and which are $\mathscr{N}-$position games.*

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Problem 88. *For the $\mathscr{N}-$position games in the previous Problem, give the move that the first player should make to turn the game into a $\mathscr{P}-$position.*

3.3 Justification of the Sprague-Grundy Theorem

In this section we'll explore the logic behind some theoretical results which will allow us to understand the Sprague-Grundy Theorem, which connects the game of nim with all other impartial games which satisfy the ending condition and are played under normal play. The problems in this section will be a little more abstract than in other sections and will not pertain to a single game or kind of game, but to games in general.

Definition 89. *We say two games G and G' are* equivalent *if $G+H$ and $G'+H$ have the same outcome class for every game H. We write $G \approx G'$ if G and G' are equivalent.*

Problem 90. *Show that $\approx$ is an equivalence relation. That is, show the following three statements.*

1. *For any game G, we have $G \approx G$.*
2. *For any games G and H, we have $G \approx H$ implies $H \approx G$.*
3. *For games G_1, G_2, and G_3, if $G_1 \approx G_2$ and $G_2 \approx G_3$, then this implies $G_1 \approx G_3$.*

Problem 91. *Show that if B and G are both $\mathscr{P}-$position games, then so is $B+G$.*

Problem 92. *Show that if H is a $\mathscr{P}-$position and H' is an $\mathscr{N}-$position, then $H+H'$ is an $\mathscr{N}-$position.*

Problem 93. *Show that if G and G' are both $\mathscr{P}-$position games, then they are equivalent.*

Problem 94. *Show that if G is any game and B is a $\mathscr{P}-$position game, then $B+G \approx G$.*

Problem 95. *Show that if $G+G'$ is a $\mathscr{P}-$position, then G and G' have the same outcome class.*

Problem 96. *Prove the following statement. If $G+G'$ is a $\mathscr{P}-$position, then $G \approx G'$.*

Problem 97. *Prove the converse to the previous Problem. That is, show that if $G \approx G'$, then $G+G'$ is a $\mathscr{P}-$position.*

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Theorem 98 (Sprague-Grundy)**.** *Every impartial game G that satisfies the ending condition played under normal play is equivalent to a game of nim with a single pile of n beans, where $n = \mathscr{G}(G)$. In other words, if the options for G are $G_1, \cdots, G_k$, then $n = \text{mex}\{\mathscr{G}(G_1), \cdots, \mathscr{G}(G_k)\}$.*

The following Problems walk through a proof of Theorem 98.

Problem 99. *Suppose G has options $G_1, G_2, \cdots, G_k$. Further, suppose G_1 is equivalent to N_1 (nim with a single pile of n_1 beans), suppose $G_2 \approx N_2$ (where N_2 is nim with a single pile of n_2 beans), etc. Let G' be the game with options $N_1, N_2, \cdots N_k$. Show that $G + G'$ is a $\mathscr{P}-$position.*

Problem 100. *Using the notation from Problem 99, show $G \approx G'$.*

Problem 101. *Let $n = \text{mex}\{n_1, n_2, \cdots, n_k\}$. Let N be nim with a single pile of n beans. Show $G' + N$ is a $\mathscr{P}-$position.*

Problem 102. *Show $G' \approx N$ and explain why this means $G \approx N$.*

Problem 103. *Explain how the previous problems constitute a proof of Theorem 98.*

To conclude this chapter, we note that we spent so much time looking at nim because *every* impartial game is really nim in disguise[7]!

[7] I hope you understand how amazing this is.

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Chapter 4

Impartial Games

In this chapter, we'll explore many impartial games. According to the Sprague-Grundy Theorem, we can fully understand the game play if we know how to translate games into nimbers, which is just an exercise in using the mex function.

4.1 Back to Take Two

Problem 104. *In the game Take Two, calculate the Grundy value of a single heap of n beans, for any positive integer n.*

Problem 105. *In the game Take Two, calculate the Grundy value of games consisting of two heaps of m and n beans, for positive integers m and n.*

Problem 106. *Use the results of the previous two problems to develop a theory of play for the game Take Two with one or two piles*[1]*. Is this consistent with what you discovered earlier in the course?*

Problem 107. *You may have had trouble with Problems 14 and 15 before. Do these problems again now that you are armed with the Grundy function.*

4.2 Back to Jordan

Problem 108. *Do Problems 104 – 107 for the game of Jordan.*

4.3 Kayles

In this section, we examine the game of Kayles, sometimes called *Rip Van Winkle's Game*. Because it is an impartial combinatorial game, we know that it's essentially nim.

[1] Another way to write this question would be to ask you to determine the outcome class of a Take Two game consisting of two or fewer piles, together with a strategy for how to play.

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The game of Kayles

Setup. Several bowling pins are set up on integer spaces on a number line, at most one pin per number. On a player's turn, he can bowl so as to knock down exactly one or two of the pins (no more, no fewer). He can only knock down two pins if they occupy consecutive interger spaces on the number line. Pins that are knocked down are removed.

Play. One player starts by bowling down one or two pins. We assume the players have perfect precision when bowling, so they always knock down the pin(s) they seek. Play then passes to the other player.

Conclusion. The first player who is unable to make a move is the loser. In Kayles, this occurs when all of the pins have been knocked down.

Problem 109. *Without computing nim values, describe the outcome class and strategy for Kayles when there is a single row of n pins with no empty spaces between them.*

Problem 110. *Any game of Kayles can be thought of as the sum of Kayles games which consist of single rows of pins with no empty spaces between them. Therefore, it is important to know the nim values of a single row of n pins*[2]*. Calculate these values for $n \leq 12$.*

Problem 111. *Give an example of two different sized rows which, when added together, yield a second player win game. Explain the second player's strategy in your example.*

4.4 Subtraction games

A generalization of nim can be played with similar, but more restrictive rules. In the definition of the game below, A is a nonempty set of positive integers. The set may be finite or infinite.

[2]Because it is important, you should calculate these values.
[3]Recall that for nim, a player can take any number of beans from a single pile on her turn.

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The Subtraction Game $S(A)$

Setup. There are several piles of beans. Each pile has several beans in it.

Play. One player starts. On a player's turn, she may take a beans from a single pile, where a is a number in the set A[3]. Play then passes to the other player.

Conclusion. The first player who is unable to make a move is the loser.

Problem 112. *Discuss*[4] *the game when* $A = \{1\}$. *The game* $S(\{1\})$ *is called She-Loves-Me-She-Loves-Me-Not.*

Problem 113. *Discuss the game when* $A = \{1,2\}$[5].

Definition 114. *For a game played with heaps of beans, we call the sequence* $\{g_n\}$ *the* nim-sequence *for the game if the Grundy value of a single heap of n beans is* g_n.

Problem 115. *Give the nim sequence of the subtraction game* $S(\{2,4,7\})$. *Is it ultimately periodic*[6]*?*

4.5 Chomp

Here's a fun game.

[4]By "discuss", I would like you to give the relevant Grundy values, explain who wins this game, give the strategy, and describe any other interesting features of the game.

[5]Hint: have we seen this game before?

[6]For many games, after a while, these sequences become periodic. That means that they begin to repeat themselves eventually, getting stuck in a pattern.

[7]Think of one of those really big Hershey bars.

[8]Thus, all sections to the south, east, and southeast of the selected section are eaten.

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Chomp

Setup. Players are to eat a large chocolate bar which is scored into several rows of several rectangular sections[7]. The top left section of the chocolate bar has been poisoned.

Play. One player starts. On a player's turn, he must choose and eat any unpoisoned section of the chocolate bar, provided he also eats all of the remaining sections below and to the right of it[8]. Play then passes to the other player.

Conclusion. The first player unable to move is the loser. In Chomp, this occurs when the only section of chocolate remaining is the poisoned one.

Problem 116. *Using the theory we've developed, give a mathematical analysis of a game of Chomp with a* 5×6 *grid of chocolate.*

Problem 117. *Pick another sized grid of chocolate and do an analysis of the Chomp game for that.*

Problem 118. *Is it possible to play Chomp on a bar of chocolate that has infinitely many rows or infinitely many columns or both? Does the Sprague-Grundy Theorem apply?*

4.6 Other impartial games

There are dozens upon dozens of impartial games. Some of them include:

- Green Hackenbush
- Grundy's game
- Dawson's chess
- Lasker's nim
- Officers
- The ruler game
- Moebius
- Trebelcross
- Antonim

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Problem 119. *Pick an impartial game not yet discussed in class*[9]*. Explain to the class how the game is played, and give some of the mathematical analysis. Give at least one example of a* $\mathscr{P}-$*position and one example of an* $\mathscr{N}-$*position. Calculate some of the Grundy values.*

[9] You can pick one of the ones listed here, you can look up one on your own, or you can ask me for suggestions.

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Chapter 5

Introduction to Partisan Games: Hackenbush

5.1 Numbers: Zero

Now, we'll start talking about partisan games. Here, "partisan" is the opposite of "impartial".

Definition 120. *A combinatorial game is* partisan *is each player has a distinct set of available moves from a given position.*

You already know several partisan games: chess and checkers (each player is playing a different color) and tic-tac-toe (each player can move in the same spaces but is using a different mark) are some examples. This contrasts to games like nim or dots-and-boxes, where the set of available moves from a position does not depend on which player is moving. Recall the definition of the word *option* as it applied to impartial games[1]. Since now each player have different options, we'll have to distinguish them. The two players in these notes will be called Lefty and Rita, or sometimes Left and Right. Lefty's options from a given position are known as the position's *left options*. You can guess what *right options* means.

Definition 121. *A partisan game G with left options $G_1^L, G_2^L, \cdots$ and right options $G_1^R, G_2^R, \cdots$ will be given the notation $G = \{G_1^L, G_2^L, \cdots | G_1^R, G_2^R, \cdots\}$.*

From this point forward, we will assume all of our games satisfy the *ending condition* from Definition 77. We will also assume *normal play* (see Definition 78).

Definition 122. *There are four* outcome classes *for partisan games:*

- *The player who goes second has a winning strategy, regardless of whether that is Left or Right;*
- *Left has a winning strategy, regardless of who goes first;*

[1]This means that if you don't remember it, you should look it up now.

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- *Right has a winning strategy, regardless of who goes first;*
- *the first player to move has a winning strategy, regardless of whether that is Left or Right.*

Problem 123. *Explain why these are the only possible outcome classes.*

As a remark, the definition of *equivalence* from Definition 89 still applies, but now we have to check four outcomes classes rather than two. Recall that impartial games all had values we called nimbers. Some partisan games have values we call *numbers*. Calling them numbers suggests that their arithmetic properties behave more normally than nimbers. We now work to define them, starting with zero.

Definition 124. *The game* $\{\ |\ \}$ *is assigned the number value*[2] *0. Here and throughout, we will write* $\{\ |\ \} = 0$ *to mean "the game with no left options and no right options has value 0". Any game which is equivalent to* $\{\ |\ \}$ *is also assigned the number value 0.*

Notice in the previous definition we are using the equals sign between the notation for a game and its value. This is standard practice when dealing with partisan games.

Problem 125. *Show that every partisan game in which the second player has a winning strategy has value 0.*

5.2 A first example: Hackenbush

To illustrate our partisan game examples, we introduce a really fun game. This is the "normal" version of Hackenbush. When discussing its variants, we may refer to this as *Red-Blue* Hackenbush.

[2]Notice this is the game in which neither player has any options.
[3]Remember that the next player will remove a stick of the opposite color.

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Hackenbush

Setup. Red and blue sticks are arranged end-to-end, so that each stick has a path to the ground via other sticks.

Play. One player starts. On a player's turn, she must remove one stick of her color. Typically we refer to the player removing the blue sticks as Lefty or Left and the player who removes the Red sticks is referred to as Rita or Right. Upon removal, any other sticks that no longer have a path to the ground via other sticks are also removed. Play then passes to the other player[3].

Conclusion. The first player who is unable to move loses. Here, that happens when all sticks of one color have been eliminated.

Here we should note that Hackenbush configurations can have a wide variety of shapes. The only restrictions are that the sticks are placed end-to-end (no sticks meeting each other in the middle), the sticks are all either red or blue, and that each stick has a path the ground via other sticks (no stick is hovering in the air). There might be many sticks touching the ground or only one. All of the sticks touching the ground might be the same color or they might not be. The sticks might make loops on themselves (if you follow end-to-end you get back to where you started) or they may not. There are a lot of possibilities, and the configurations are fun to draw.

Problem 126. *Give two examples of Hackenbush games which have the number value 0.*

Problem 127. *Give two examples of Hackenbush games which don't have the number value zero and which you don't think are equivalent. Explain why you don't think they are equivalent.*

Problem 128. *If numbers (the way we're about to define them as they assign value to a game) make any sense, make an educated guess as to the number value of a few very simple Hackenbush games. Give some reasoning for your answer.*

5.3 Numbers other than zero

Next, we define games with nonzero integer values.

Definition 129. *The game $\{\ \{|\}\ |\ \}$ is defined to have the value 1. This is the game where left's only option is to move to the game with no option, and right has no options at all. Because the game $\{\ |\ \}$ has been previously defined to be 0, the game with value 1 can also be written as $\{0|\ \}$. Any game equivalent to this game is also said to have value 1.*

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Problem 130. *Give an example of a Hackenbush game with the value of 1.*

We can keep going like this, defining that a game has number value 2 if it has option $\{1|\ \}$ or is equivalent to this. Inductively, we make the following definition.

Definition 131. *For x a nonnegative integer, the game $\{x|\ \}$ is said to have value $x+1$, and this is typically written $\{x|\ \} = x+1$. Any game equivalent to $\{x|\ \}$ is also said to have the value $x+1$.*

Problem 132. *Give example of Hackenbush games with the values of 2, 3, 4, and 5. In each of the examples you give, use the bracket notation to justify why your example has the number value you claim.*

Problem 133. *Give an example of a Hackenbush game with value 3 which is different than the example you gave in the previous problem. Again, use the bracket notation to justify the number value you claim.*

Problem 134. *Prove or disprove: If each left option for a game G is equivalent to a left option in a game G' and vice versa*[4]*, and each right option in G is equivalent to a right option in G' and vice versa, show that $G \approx G'$.*

Problem 135. *Prove that $\{\ |\ \} + \{\ |\ \} = \{\ |\ \}$. What this statement means is that the games on the left and the right side have the same number value.*

Problem 136. *Prove that for any game G, $G + \{\ |\ \} = G$.*

Problem 137. *Show that $\{\ \{|\}\ \big|\ \} + \{\ \{|\}\ \big|\ \} = \{\ \{0|\}\ \big|\ \}$. Yeah, I went there*[5]*.*

Problem 138. *Show that if G and H are positive integers*[6]*, then the value of $G+H$ is the value of G plus the value of H.*

Definition 139. *The* negative *of a game G is the game which results by reversing the roles of Left and Right*[7]*. We denote the negative of G by $-G$.*

Problem 140. *Show that $G - G = 0$. (Really, this means $G + (-G) = 0$.)*

Definition 141. *If G is a positive number, define the value of $-G$ to be the negative of the value of G.*

Problem 142. *Give an example of a (not necessarily Hackenbush) game with value -4.*

Problem 143. *Show that, for $x > 0$, the game $\{\ |-(x-1)\}$ has value $-x$.*

[4]By *vice versa*, I mean that every left option in G' is equivalent to one in G.

[5]Did you notice that I just asked you to prove 1+1=2?

[6]This verbiage is typical. We say "G is a positive number" as shortspeak for "G has a value which is a positive number." Notice this is another example of equating games with their values.

[7]That is, the Right's legal moves become Left's and vice versa.

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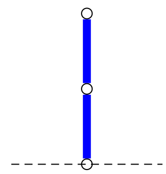

Figure 5.1: For Problem 150; two blue sticks

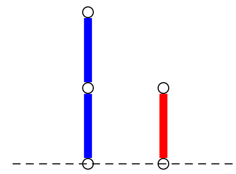

Figure 5.2: For Problem 151; two blue sticks on the left, one red stick at right

Problem 144. *If $x > y$ and both are positive integers, show that* $\{x|\ \} + \{\ | -y\} = \{x - y - 1|\ \}$.

Problem 145. *If $y > x$ and both are positive integers, show that* $\{x|\ \} + \{\ | -y\} = \{\ | -(y-x)+1\}$.

Problem 146. *Prove or disprove: If G and H have integer number values, then the value of $G+H$ is the value of G plus the value of H.*

Problem 147. *If a game is positive*[8]*, which player (if any) has a winning strategy? What if the game is negative?*

Problem 148. *For x a positive integer, show $\{x, 0|\ \} = x+1$. Note: this is the first time we've seen a game with more than one left option.*

Definition 149. *We say a game G has value $\frac{a}{b}$ if*

$$\underbrace{G+G+\ldots+G}_{b \text{ copies}} + \{\ | -a+1\} = 0$$

when a and b are positive integers.

5.4 Return to Hackenbush

Problem 150. *Find the number value of the Hackenbush game in Figure 5.1.*

Problem 151. *Find the number value of the Hackenbush game in Figure 5.2. Prove your result using the definition of number.*

Problem 152. *Find the number value of the Hackenbush game in Figure 5.3.*

Problem 153. *Find the number value of the Hackenbush game in Figure 5.4.*

[8]Recall that when a game has a number value, we'll often say the game *is* a number, so statements about the game being positive, negative, etc. make sense.

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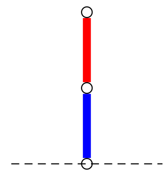

Figure 5.3: For Problem 152; a red stick atop a blue stick

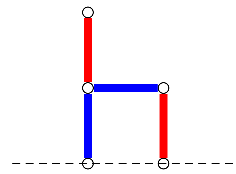

Figure 5.4: For Problem 153; a chair with red back and front leg, blue seat and back leg

5.5 Simplicity rule

This section pertains to any partisan game, not just Hackenbush. Our goal here is to be able to look at the number values of all of the options and be able to determine the number value of the game itself. Let's see if we can figure out how this works.

Problem 154. *Suppose* $x_1, x_2, \ldots, x_m$ *and* $y_1, y_2, \ldots, y_m$ *are numbers. Explain why the game* $\{x_1, x_2, \ldots, x_m | y_1, y_2, \ldots, y_n\}$ *is equivalent to the game* $\{x_i | y_j\}$*, where* x_i *is the maximum value of* x_1 *through* x_m *and* y_j *is the minimum value of* y_1 *through* y_n*.*

Problem 154 tells us that *if all of our options are numbers*[9], then we can effectively ignore all but the best options for left and right. Lefty will make the best option for him: the largest (most positive) one. Rita will make the best option for her: the smallest (most negative) one. Any other (number) option we can simply ignore. From here on out we'll often not bother to write options that are irrelevant in this way; such options are called *dominated options* and are never used in optimal game play.

Problem 155. *Find the value of* $\{0|1\}$*. Hint: Write out the game in Problem 152 in bracket notation.*

Problem 156. *Find the value of* $\{1|2\}$*. Draw a Hackenbush configuration which has the same value as this game.*

Problem 157. *Find the value of* $\{-4|2\}$*.*

Problem 158. *Find the value of* $\{1|3\}$*.*

Problem 159. *Find the value of any of the options of the game in Problem 153.*

OK, so finding the pattern here can be a bit difficult. We'll now present the simplicity rule for finding the number value of games. First, we make a comment about which games are numbers.

Theorem 160. *A game is a* number *if and only if it is of the form* $\{x|y\}$ *with* $x < y$ *(and* x *and* y *are numbers). In other words, its options are numbers*

[9] Soon we will see games where not all of our options are numbers!

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and the value of its largest left option is less than the value of its smallest right option.

Theorem 161 (Simplicity rule)**.** *The number value of the game $\{x|y\}$ with $x < y$ is the simplest number between x and y. This means that $\{x|y\} = n$ where n is a number such that no option of n is between x and y.*

The simplicity rule says how to calculate values, but the definition of "simplest" is hard to understand. Here are some guidelines.

- 0 is the simplest number.
- If the interval (x, y) contains an integer, then the simplest number between x and y is the integer in the interval (x, y) with the smallest absolute value.
- If the interval (x, y) contains a half integer but not an integer, then it is the simplest number between x and y.
- If the interval (x, y) contains a quarter integer but not an integer or half integer, then it is the simplest number[10] between x and y.

Problem 162. *Draw a hackenbush game with value $\frac{1}{4}$.*

Problem 163. *Find the value of $\{1\frac{1}{4}|2\}$*

Problem 164. *Find the value of $\{1\frac{3}{8}|2\}$.*

Problem 165. *Find the value of $\{1\frac{5}{16}|1\frac{7}{16}\}$.*

[10] This pattern continues with eighth integers, etc.

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Chapter 6

General partisan games

So far in our study of partisan games, we've talked about numbers and we've played Hackenbush. Hackenbush is a great game because all of its configurations are numbers. However, that is not true of all partisan games. Let's generalize what we've done and play a few more games along the way.

6.1 Positive, negative, and fuzzy games

Definition 166. *We say a game G is* positive *and write $G > 0$ if Left has a winning strategy, regardless of who moves first. Similarly, a game G is* negative *(and we write $G < 0$) if Right has a winning strategy, regardless of who moves first. If the second player to move has a winning strategy, regardless of whether that is Left or Right, then we say $G = 0$. If the first player to move has a winning strategy, regardless of which player that is, we say the game is* fuzzy*, or* confused with zero*, and write $G||0$.*

Problem 167. *Label each of the following games as positive, negative, zero, or fuzzy. Justify your responses.*

1. $\{-2|-1\}$
2. $\{-2|1\}$
3. $\{-1|-2\}$
4. $\{0|0\}$
5. $\{\,|-3\}$
6. $\{-3|\,\}$

Definition 168. *The game $\{0|0\}$ arises often enough that it has a special name. We denote the game $\{0|0\}$ as $*$, and say it aloud as "star".*

Problem 169. *Is $*$ positive, negative, zero, or fuzzy?*

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Problem 170. *If $G > 0$ and $H > 0$, what can you say about*[1] $G+H$*? Prove your claims.*

Problem 171. *If $G > 0$ and $H < 0$, what can you say about $G+H$? Justify your claims.*

Problem 172. *If $G > 0$ and $H = 0$, what can you say about $G+H$? Justify your claims.*

Problem 173. *If $G > 0$ and $H||0$, what can you say about $G+H$? Justify your claims.*

Problem 174. *If $G = 0$ and $H||0$, what can you say about $G+H$? Justify your claims.*

Problem 175. *Use the bracket notation to describe the game of nim with a single pile of 2 beans. Classify this as positive, negative, zero, or fuzzy.*

In fact, because a nim pile of one bean is $\{0|0\} = *$, we have motivation for our previous notation of $*1$ (one star) for the game as an impartial game. The star that arises here is akin to the stars used for impartial games.

Problem 176. *Suppose $G||0$. Is it necessarily true that $G = *$?*

Problem 177. *Prove or disprove: $\{*,0|*,0\} = *$.*

6.2 Domineering

In this section, we meet a new game. I'll call it Domineering, but some people call it Crosscram.

Domineering

Setup. The game is played on a checkerboard[2]. Both players have dominoes which are sized so that they cover exactly two adjacent squares on the checkerboard.

Play. On Left's turn, he places one domino, oriented North-South, on the checkerboard so that it covers two empty spaces (spaces not already covered by a domino). On Right's turn, she places one domino, oriented East-West, on the checkerboard so that it covers two empty spaces.

Conclusion. The first player who is unable to make a move is the loser.

[1] In particular, must it be positive? Must it be negative? Must it be either negative or fuzzy? Could it be any of those?

[2] The checkerboard may be any size or shape, and doesn't need to be colored.

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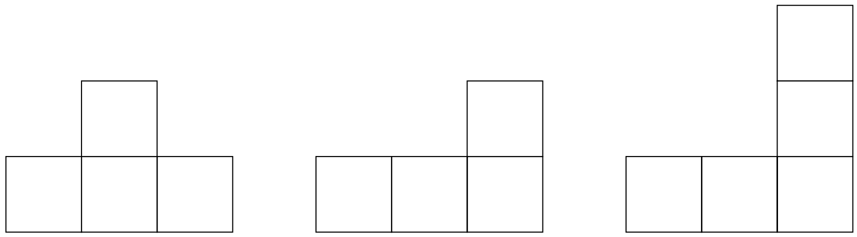


Figure 6.1: Domineering boards for Problems 178 – 180.

Problem 178. *Calculate the value of the Domineering position shown in the left panel of Figure 6.1.*

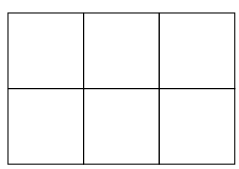
Problem 179. *Calculate the value of the Domineering position shown in the center panel of Figure 6.1.*

Problem 180. *Calculate the value of the Domineering position shown in the right panel of Figure 6.1.*

Problem 181. *Calculate the value of the opening position of Domineering on a 3×3 checkerboard. Explain who has a winning strategy, and describe the strategy.*

6.2.1 Switches

Problem 182. *Calculate the value of the domineering position given below.*



Problem 183. *Denote the domineering position from the previous problem as D. Suppose G is a number; call its value v. For which values of v is $D+G>0$? For which values of v is $D+G<0$? For which values of v is $D+G \mid\mid 0$?*

Definition 184. *A* switch *value is a game of the form $\{x|y\}$ with $x \geq y$.*

Problem 185. *Suppose G is a number and H is a switch. It's your turn to play in $G+H$. In which game should you move? Why?*

Problem 186. *Generalize the result of Problem 183 to the definition of switch.*

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Chapter 7

More partisan game examples

7.1 Toads and Frogs

Let's play another game, this time with amphibians.

Toads and Frogs

Setup. A $1 \times n$ strip of squares is populated with toads and frogs, with each square containing at most one animal.

Play. On Left's turn, he may make one of two moves. He may move any toad one square to the right, provided that square is empty. Or, he may move any toad two squares to the right if the square immediately to the right of the toad is occupied by a frog and the target square is empty[1]. On Right's turn, she may move any frog one square to the left (if that square is empty) or she may move a frog two squares to the left if the square immediately to the left is occupied by a toad.

Conclusion. The first player who is unable to make a move is the loser.

We'll display the Toads and Frogs position in a chart like the one below[2].

T		F

Problem 187. *What is the outcome class on the Toads and Frogs position above?*

Problem 188. *Express the Toads and Frogs position above in bracket notation.*

[1] In this way, the toad "jumps" the frog.
[2] 'T' stands for toad, 'F' stands for frog.

Problem 189. *Use bracket notation to describe the Toads and Frogs position below.*

T		*T*	*F*	*F*

Problem 190. *Calculate the value of your favorite Toads and Frogs position.*

7.2 Up and down

Problem 191. *Show that the game* $\{0|*\}$ *is positive.*

Problem 192. *Show that the game* $\{0|*\}$ *is less than* $1/2$.

Problem 193. *Show that the game* $\{0|*\}$ *is less than* $1/4$.

In fact, it can be shown that $\{0|*\}$, even though it is positive, is less than any positive number.

Definition 194. *The game* $\{0|*\}$ *is called* up*, and is usually abbreviated* $\uparrow$. *Similarly, the game* $\{*|0\}$ *is called* down *and abbreviated* $\downarrow$.

Problem 195. *Show* $\{\uparrow | \downarrow\} = *$.

Problem 196. *Prove or disprove:* $\uparrow + * = *$.

Problem 197. *Which of the Toads and Frogs positions from section 7.1 is* $\uparrow$*?*

7.3 Cutcake

I'm hungry[3]. Somebody made a cake, we just need somebody to cut it.

[3] We haven't had a game with beans for a while.

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Cutcake

Setup. An $m \times n$ grid is put on top of a cake. These grid lines will serve as a guide for where the cake is to be cut.

Play. On Left's turn, he must cut a section of the cake along a north-south grid line. On Right's turn, she must cut a section of the cake along an east-west grid line. Once a grid line has been cut once, it cannot be cut again. When cutting, the knife does not pass over grid lines that have already been cut: cuts are to be made only in a single section (a newly formed piece of the cake, after a cut has been made) of the cake.

Conclusion. The first player who is unable to make a move is the loser.

Problem 198. *Calculate the value of a* 2×4 *cake*[4].

Problem 199. *Calculate the value of a* 3×3 *cake.*

7.4 Maundy Cake

On Maundy Thursday, a special cake cutting game is played by Lefty and Rita. Would you like to join?

Maundy Cake

Setup. An $m \times n$ grid is put on top of a cake. These will serve as a guide for where the cake is to be cut.

Play. On Left's turn, must cut a single section of the cake along any number of north-south grid lines such that the resulting pieces are all of equal size. On Right's turn, she must cut a single section of the cake along any number of east-west grid lines such that the resulting pieces are all of equal size. Once the cake has been cut along a grid line, the cake then becomes divided into sections along that cut. As indicated, subsequent cuts are made in a single section of the cake.

Conclusion. The first player who is unable to make a move is the loser.

Problem 200. *Calculate the value of a* 2×4 *cake.*

Problem 201. *Calculate the value of a* 3×3 *cake.*

[4]This means there are three north-south lines and one east-west line on the cake, dividing it into 8 pieces.

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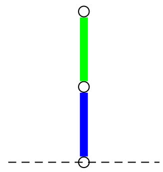

Figure 7.1: For Problem 202; a green stick atop a blue stick

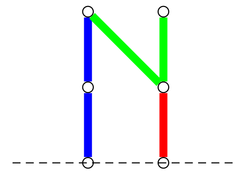

Figure 7.2: For Problem 203; two blue sticks at left, a red stick on bottom right, two green sticks at top right

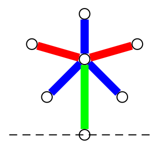

Figure 7.3: For Problem 205; a green stem, three blue petals and two red petals

7.5 Hackenbush Hotchpotch

Here's a game galled Hackenbush Hotchpotch, also known as Red-Blue-Green Hackenbush.

Hackenbush Hotchpotch

Setup. Red and blue and green sticks are arranged end-to-end, so that each stick has a path to the ground via other sticks.

Play. One player starts. On Lefty's turn, he must take one stick, which can be blue or green. On Rita's turn, she must take one stick, which can be red or green. Upon removal, any other sticks that no longer have a path to the ground via other sticks are also removed. Play then passes to the other player.

Conclusion. The first player who is unable to move loses.

Problem 202. *Find the value of the Hackenbush Hotchpotch game in Figure 7.1.*

Problem 203. *Find the value of the Hackenbush Hotchpotch game in Figure 7.2.*

Problem 204. *Draw your own pretty Hackenbush Hotchpotch picture and calculate its value.*

Problem 205. *Consider the game in Figure 7.3; call it G. Is G positive, negative, or fuzzy? Is $G+G$ positive, negative, or fuzzy?*

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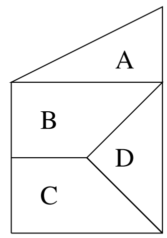

Figure 7.4: For Problems 206 – 210

7.6 Snort

A game introduced by Simon Norton has been called *Snort*[5].

Snort

Setup. A parcel of land is divided into several fields into which bovines can be placed.

Play. Lefty likes to buy bulls and Rita likes to buy cows. They have inherited this parcel of land and wish to make the fields into grazing pastures for the bovines. They take turns picking which field they would like for their animals. That is, on Lefty's turn, he picks a field to populate with bulls and on Rita's turn she selects a field to populate with cows. They agree that it would not be wise to put cows and bulls in adjacent fields, so this is not allowed.

Conclusion. The first player who is unable to move loses. In this case, this happens when one player cannot find a field which is not adjacent to a field being used by the opposite player. (Touching only at one point is OK)

Problem 206. *If Lefty placed some bulls in field A in Figure 7.4, what is the bracket notation for the resulting game of Snort?*

Problem 207. *Let G be the game*[6] *in Problem 206 and let H be the game of Hackenbush with two red sticks. What is the outcome class of $G+H$?*

Problem 208. *As in the previous problem, let G be the game in Problem 206, but this time let H be a Hackenbush game with three red sticks. What is the outcome class of $G+H$?*

Problem 209. *Again, let G be the game in Problem 206. Now let H be a Hackenbush game with value $\frac{1}{2}$. What is the outcome class of $G+H$?*

[5]So, if you invent a game and your name can be used against you, it's best to give it a name yourself rather than letting someone else name it after you.

[6]Remember, Lefty already has bulls in field *A*.

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Problem 210. *If Lefty places his bulls in field B (instead of field A) in Figure 7.4, what is the value of the resulting game?*

Problem 211. *Explain why the following statement is true or why it is false: all games of Snort are either zero or fuzzy.*

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