MATHEMATICAL NOTES

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A CURIOUS NIM-TYPE GAME

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A set of mn objects is laid out in an m by n rectangular array. We denote by (i,j) the object in row i, column j. The first player P_{I} selects an object (i_1,j_1) and then removes all objects (i,j) such that $i \geqq i_1$ and $j \geqq j_1$. In other words, if i increases upward and j increases from left to right, then P_{I} removes a northeast "quadrant." Player two, P_{II}, now picks (i_2,j_2) from among the remaining objects and removes all (i,j) such that $i \geq i_2$, $j \geq j_2$. The play then reverts to P_{I} and continues in the same way until all objects have been removed. The player making the last move loses. Thus the object of the game is to make your opponent pick up $(1,1)$.

There are some trivial special cases of the game.

CASE A: P_{I} wins the $2 \times n$ $(m \times 2)$ game by selecting $(2,n)((m,2))$. Then, whatever P_{II} does, P_{I} moves so as to leave a "position" in which there is one more object in row (column) 1 than in row (coumn) 2. The reader will easily see that this is always possible and winning.

CASE B: P_{I} wins the $m \times m$ game by selecting $(2,2)$. From then on he "symmetrizes." Whenever P_{II} chooses $(1,j)$ he chooses $(j,1)$, etc. Again this is easily seen to win.

The above are the only two cases in which general winning strategies are known. The thing which makes the game interesting, however, is the following

THEOREM. *For all m and n the game is a win for* P_{I}.

The proof of this fact is typical of something which occurs quite often in game theory in that it is completely nonconstructive. Although it establishes the existence of a winning strategy for P_{I} it is of absolutely no use in finding such a strategy. Here is the argument. There are two possibilities.

CASE 1: P_{I} has a winning strategy in which his first move is to select (m,n).

CASE 2: If P_{I} selects (m,n) he loses. Then there must be a response (i_2,j_2) by P_{II} which wins for P_{II}. This means that the position of the game after P_{II}'s move is a loss for the player who must then move, in this case P_{I}. The point is, however, that P_I could have handed this position to P_{II} if he had himself chosen (i_2,j_2). Hence (i_2,j_2) is a winning first move for P_I, and the assertion is proved.

The above argument is reminiscent of the well known one which asserts that games like tic-tac-toe cannot be a win for player II (it applies, for instance, to the unsolved Go Moku which is tic-tac-toe, 5 in a row on a 19 by 19 board.) It goes like this: suppose the game was a win for P_{II}: Then let P_I make any first move and then pretend in his mind he did not make it, and from then on behave in accordance with the alleged winning strategy for P_{II}. This will always be possible unless at some point this strategy requires him to move onto the square he chose initially. In that case, he again makes an arbitrary move. We see then that the alleged winning strategy for P_{II} is also available to P_I, but by definition of winning there cannot be winning strategies for both players. Contradiction. (Note, however, that the tic-tac-toe proof as given here is a proof by contradiction while the proof of our theorem is direct, which points up the interesting fact that a non-constructive proof is not necessarily a proof by contradiction.)

It may be of interest to observe that both Nim and this game (Gnim? Gnome?) are special cases of the following general class of games: Let S be any partially ordered set. A player moves by choosing some element of S and removing all elements greater than or equal to it, and the player moving last loses. Nim corresponds to the special case where S is the *sum* (disjoint union) of a finite number of totally ordered sets; Gnim is the case where S is the product of two totally ordered sets. The argument showing that Gnim is a win for P_I applies to any set S which has a *largest* element, thus, for example, the product of any number of totally ordered sets, but of course not to Nim.

The above is essentially all the theoretical information I have about this game (I shall give one further special result at the end.) However, with the aid of a computer some rather intriguing empirical results have been obtained. The $3 \times n$ game has been completely analyzed for $n \leqq 100$ and in all cases it has turned out that there is only one winning first move for P_I. This is also the case for 4×5 and 4×6 and of course Cases A and B discussed in the beginning.

Question. Is the winning first move unique for all m and n? (See note **"Added in proof."**)

The diagrams below give the winning first move in the $3 \times n$ game for $2 \leqq n \leqq 12$.

```
×      ×        ××        ×××          ×××             ××××              ××××
××     ×        ××        ×××××        ×××             ×××××××           ××××
××     ×××      ××××      ×××××        ××××××          ×××××××           ××××××××
2      3        4         5            6               7                 8

××××××          ×××××             ××××××               ××××××××
×××××××××       ×××××             ××××××               ××××××××××××
×××××××××       ××××××××××        ×××××××××××          ××××××××××××
9               10                11                   12
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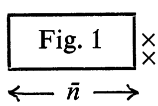
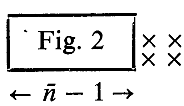
There are of course two types of moves depending on whether one takes the initial "bite" from the top row only or the top two rows. It turns out that roughly 58 percent of the moves are of this second type, as for the case $n = 3, 4, 6, 8, 10, 11$, and 42 percent of the first, e.g., $n = 2, 5, 7, 9, 12$. In general the length of the bite appears to increase with n. In fact for all $n \leqq 170$ there is only one counterexample. For $n = 87$ the bite size is 37, while for $n = 88$ the bite size is 36 (both of these are two-row bites). Phenomena like this lead one to believe that a simple formula for the winning strategy might be quite hard to come by.

We close with a conjecture: *it is never optimal to select* (m, n) *on the first move except when* $m = 2$ *or* $n = 2$. We shall prove this for the case $m = 3$ as a further illustration of the type nonconstructive argument one uses. This one requires an argument by contradiction.

First observe that in the 3×3 game $(3,3)$ is losing (since $(2,2)$ wins). Now assume that $(3, n)$ is losing for all $3 \times n$ games up to $\bar{n}$ and consider the case $3 \times (\bar{n} + 1)$. The argument is best given with pictures

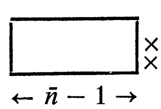
suppose 

is a winning position

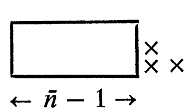
then 

must be losing,

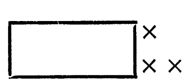
so there must be a way of going from Fig. 2 to a winning position. Now clearly no choice (i,j), $j \leqq \bar{n}$ can give a winning position for this would leave a position which P_{II} could have presented to P_I, contradicting the assumption that $(3, \bar{n} + 1)$ was winning for P_I. The only possible choices are therefore either $(\bar{n}, 1)$ or $(\bar{n}, 2)$, but $(\bar{n}, 1)$ leaves



which is losing by the induction hypothesis and $(\bar{n}, 2)$ leaves



after which P_{II} can play (3,1) leaving



which is losing for the $2 \times (n + 1)$ game of Case A, so the proof is complete.

One can prove any number of special results of this sort by similar arguments.

For example, for $n > 4$ it is never winning to choose $(2, n-1)$. For $n > 5$ it is never winning to choose $(3, n-1)$, and presumably some general inequalities exist showing that for large rectangles the bites cannot be too small. I expect the problem of finding explicit winning strategies may be hopeless, but I should think one might find a way of settling questions like the uniqueness of the first move.

Finally, let me mention some generalizations. The first is to allow either m or n or both to be infinite. However, these games turn out to be rather trivial because (A) $1 \times \infty$ is a win for P_{I} (trivial), (B) $2 \times \infty$ is a win for P_{II} (a nice exercise for the reader) and (C) $m \times \infty$, $2 < m \leqq \infty$, is a win for P_{I}, as he can choose (2,1) leaving P_{II} with $2 \times \infty$. Of more interest are higher dimensional games, e.g. m by n by r in 3-space. Of course, any such finite game can be solved in a finite amount of time by, at worst, enumerating all possible strategies. The real challenge, it seems to me, are games like $3 \times 3 \times \infty$ or even $\infty \times \infty \times \infty$. ($2 \times m \times \infty$ is a win for P_{I}. Why?) These belong to an interesting class of games with the property that although every play of the game terminates after a finite number of moves there is no upper bound on the possible lengths of a play (as there is for example in chess). In particular, I don't know of any way to program a computer to find out, say, if $3 \times 3 \times \infty$ is a win for P_{I} or P_{II}.

Added in Proof: Since this article was submitted, a description of the game appeared in the column of Martin Gardner in the magazine "Scientific American," pp. 110–111, January 1973. In response to the article, K. Thompson of Bell Laboratories and M. Beeler at M.I.T. discovered by using computers that there exist games with more than one winning first move. The smallest known example is 8×10. Further it was learned that a numerical game isomorphic to this one was described by F. Schuh in an article entitled "The Game of Devisors" in Nieuw Tijdschrift voor Wiskunde, Vol. 39, pp. 299–304, 1952.

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THE POWER MEAN AND THE LOGARITHMIC MEAN

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The logarithmic mean of two distinct positive numbers x and y, defined by

$$L = L(x, y) = \frac{x - y}{\ln x - \ln y}, \text{ for all distinct } \quad x, y > 0, \tag{1}$$

is quite frequently used in some practical problems, such as in heat transfer and fluid mechanics. The power mean of two positive numbers x and y, defined by

$$M_p = M_p(x, y) = \left(\frac{x^p + y^p}{2}\right)^{1/p}, \qquad x, y > 0, \tag{2}$$

for any real number $p \neq 0$, is a generalization of the so-called "root-mean-square average." Thus