Prace Naukowe Wyższej Szkoły Pedagogicznej w Częstochowie
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Nadreálná čísla

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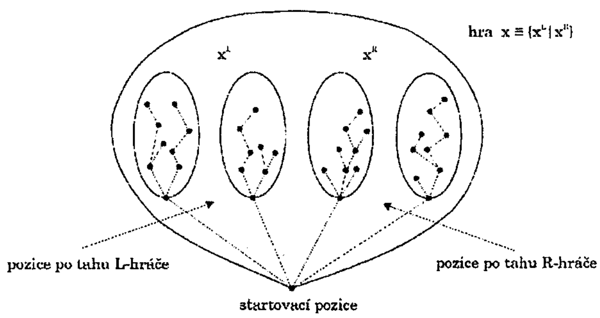
1. Ideové zdroje teorie nadreálných čísel

- Dedekind konstruuje reálná čísla z racionálních čísel tímto způsobem: Rozkládá racionální čísla do dvou množin A, B (každé číslo z A je menší než libovolné číslo z B) a tento „řez" — uspořádaná dvojice množin $\{A \mid B\}$ — je užit k definici nových čísel. Je to konstrukce směrem dovnitř, číselná množina se zahušťuje.
- Cantor (von Neumann) konstruuje ordinální čísla tak, že každé z nich chápe jako množinu čísel již dříve zkonstruovaných, např. $2 = \{0, 1\}, \omega = \{0, 1, 2, \ldots\}$, atd. Toto je konstrukce směrem ven, číselná třída se rozšiřuje.
- Některé speciální hry mezi dvěma hráči mohou být reprezentovány pomocí logických stromů:



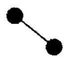
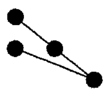
L-hráč může táhnout v každé pozici jenom nalevo, R-hráč pouze napravo. Hráči se pravidelně střídají, jestliže hráč nemá žádný možný další tah, pak prohrál.

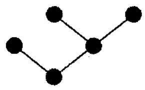
Conway chápe každou hru jako uspořádanou dojici množin již dříve zkonstruovaných her (tzv. levé subhry označujeme x^L, pravé x^R).

2. Hry jako matematické objekty

Konstrukce 1 *Jsou-li L a R dvě množiny her, pak $\{L \mid R\}$ je hra. Všechny hry jsou zkonstruovány tímto způsobem.*

Příklady 1

 $\equiv 0 \equiv \{\mid\}$  $\equiv 1 \equiv \{0 \mid\}$  $\equiv 2 \equiv \{0, 1 \mid\}$

 $\equiv \star \equiv \{0 \mid 0\}$  $\equiv -1 \equiv \{\mid 0\}$  $\equiv \uparrow \equiv \{0 \mid \star\}$

Tato konstrukce spojuje oba výše zmíněné konstrukční přístupy (Dedekindův i Cantorův), třída her se současně „zahušt'uje" i „rozšiřuje".

Definice 1 (Opačná hra) $-x \equiv \{-x^R \mid -x^L\}$

Definice 2 (Součet her) $x + y \equiv \{x^L + y, x + y^L \mid x^R + y, x + y^R\}$

Na třídě her G lze definovat i relace $>$, $<$, $\|$, $=$ tak, že mají tyto strategické interpretace:

$x > 0$	$\Longleftrightarrow$	existuje vyhrávající strategie pro L-hráče
$x < 0$	$\Longleftrightarrow$	existuje vyhrávající strategie pro R-hráče
$x \parallel 0$	$\Longleftrightarrow$	existuje vyhrávající strategie pro hráče, který táhne jako první
$x = 0$	$\Longleftrightarrow$	existuje vyhrávající strategie pro hráče, který táhne jako druhý

Příklady 2

$-\star \equiv \star \quad -\uparrow \equiv \{\star \mid 0\} \qquad \star \| 0 \quad \uparrow > 0$

$\star + \star \equiv \{\star \mid \star\} = 0 \qquad 1 + (-1) \equiv \{-1 \mid 1\} = 0$

Lze dokázat, že struktura $[G, =, +, -, 0]$ je neuspořádaná komutativní grupa.

3. Nadreálná čísla

V třídě her G je možné vymezit třídu nadreálných her čísel **No** touto definicí:

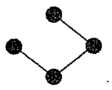
Definice 3 *Hra $x \equiv \{x^L \mid x^R\}$ je (nadreálné) číslo, právě když každá subhra x^L a x^R je číslo a platí:*

$$(\forall x)(\forall y) \quad x^L < x^R \, .$$

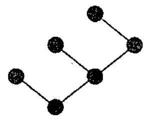
Je pak možné dokázat, že pro každé číslo $x \equiv \{x^L \mid x^R\}$ platí, že

$$(\forall x)(\forall y) \quad x^L < x < x^R \, .$$

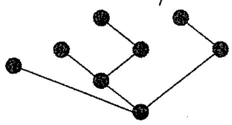
Příklady 3



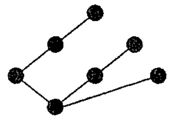
$\equiv 1/2 \equiv \{0 \mid 1\}$



$\equiv 1/4 \equiv \{0 \mid 1/2\}$



$\equiv 3/4 \equiv \{0, 1/2 \mid 1\}$



$\equiv -3/4 \equiv \{-2 \mid -1, 0\}$

Na třídě nadreálných čísel **No** je možné definovat násobení čísel a invertování čísla těmito definicemi:

Definice 4 (Součin čísel)

$$x \cdot y \equiv \{x^L y + x y^L - x^L y^L, x^R y + x y^R - x^R y^R \mid$$
$$\mid x^L y + x y^R - x^L y^R, x^R y + x y^L - x^R y^L\}$$

Definice 5 (Inverzní číslo) *Pro kladné číslo x :*

$$y \equiv \frac{1}{x} \equiv \left\{0, \frac{1 + (x^L - x)y^R}{x^L}, \frac{1 + (x^R - x)y^L}{x^R} \right|$$
$$\left| \frac{1 + (x^L - x)y^L}{x^L}, \frac{1 + (x^R - x)y^R}{x^R} \right\}$$

Je možn'e pak dokázat, že struktura $[\mathbf{No}, =, <, +, -, \cdot, 1/, 0, 1]$ je lineárně uspořádané komutativní těleso, je možné definovat odmocninu z kladných čísel a jiné další funkce. Toto těleso obsahuje všechna čísla reálná, všechna čísla ordinální, čísla nekonečně malá, atd.

Příklady 4

$$2/3 = \{0, 1/2, 5/8, 21/32, \ldots 1, 3/4, 11/16, \ldots\}$$
$$\sqrt{2} = \{1, 7/5, 1393/985, \ldots 3/2, 99/70, \ldots\}$$

$\omega \stackrel{def}{\equiv} \{0, 1, 2, \ldots \mid \}$

$\omega + 1 = \{0, 1, 2, \ldots, \omega \mid \}$

$\omega - 1 = \{0, 1, 2, \ldots \mid \omega\}$

$\omega - 2 = \{0, 1, 2, \ldots \mid \omega, \omega - 1\}$

$\omega + 1/2 = \{0, 1, 2, \ldots, \omega \mid \omega + 1\}$

$\omega - 1/2 = \{0, 1, 2, \ldots, \omega - 1 \mid \omega\}$

$\omega/2 = \{0, 1, 2, \ldots \mid \omega, \omega - 1, \omega - 2, \ldots\}$

$\omega/4 = \{0, 1, 2, \ldots \mid \omega/2, \omega/2 - 1, \ldots\}$

$\sqrt{\omega} = \{0, 1, 2, \ldots \mid \omega, \omega/2, \omega/4, \ldots\}$

$\varepsilon \stackrel{def}{\equiv} \{0 \mid 1, 1/2, 1/4, \ldots\}$

$\varepsilon/2 = \{0 \mid \varepsilon\}$

$\varepsilon/4 = \{0 \mid \varepsilon, \varepsilon/2\}$

$\varepsilon^2 = \{0 \mid \varepsilon, \varepsilon/2, \varepsilon/4, \ldots\}$

$2 \cdot \varepsilon = \mid 1, 1/2, 1/4, \ldots\}$

$3 \cdot \varepsilon = \{2\varepsilon \mid 1, 1/2, 1/4, \ldots\}$

$\sqrt{\varepsilon} = \{0, \varepsilon, 2\varepsilon, \ldots \mid$
$\mid 1, 1/2, 1/4, \ldots\}$

$1/\varepsilon = \omega$

atd.

Je velmi zajímavé, jak bohatá je struktura nadreálných čísel, jak mnoho „infinitezimálních" čísel je v okolí 0 (a tedy i v okolí libovolného čísla), jak se oproti běžným ordinálním číslům obohatila struktura „nekonečných" čísel, atd.

4. Ordinální operace na číslech

Na třídě čísel **No** lze definovat sčítání a násobení i jiným způsobem:

Definice 6 (Ordinální součet)

$$x \oplus y \equiv \{x^L, x \oplus y^L \mid x^R, x \oplus y^R\}$$

Definice 7 (Ordinální součin)

$$x \odot y \equiv \{x \odot y^L \oplus x^L, x \odot y^R \oplus x^R \mid x \odot y^L \oplus x^R, x \odot y^R \oplus x^L\}$$

Je dokazatelné, že tyto operace mají tytéž vlastnosti jako obvyklé ordinální operace (neutrální vlastnosti prvků 0 a 1, asociativnost obou operací, distributivitu pouze zleva, nekomutativnost obou operací, atd.).

Pokud parcializujeme tyto operace na třídu ordinálních čísel **On**, která je vymezena v rámci **No** definicí:

Definice 8 *Číslo x je ordinálním číslem, právě když $x \equiv \{x^L \mid\}$ a subhry x^L jsou všechna ordinální čísla menší než x,*
získáme obvyklou uspořádanou strukturu ordinálních čísel
$[\mathbf{On}, \equiv, <, \oplus, \odot, 0, 1]$.

Ordinální operace na třídě **No** pak dávají obecnější zajímavé výsledky:

$1 \oplus (-1) = 1/2$	$1 \oplus (-2) = 1/4$	$1 \oplus (-3) = 1/8$	$1 \oplus (-\omega) = \varepsilon$
$1/2 \oplus 1 = 3/4$	$1/2 \oplus 2 = 7/8$	$1/2 \oplus 3 = 15/16$	$1/2 \oplus \omega = 1 - \varepsilon$
$\varepsilon \oplus 1 = 2 \cdot \varepsilon$	$\varepsilon \oplus 2 = 3 \cdot \varepsilon$	$\varepsilon \oplus 3 = 4 \cdot \varepsilon$	$\varepsilon \oplus \omega = \sqrt{\varepsilon}$

$\omega \odot 1/2 = \omega - 1$	$\omega \odot 1/4 = \omega - 2$	$\omega \odot 1/8 = \omega - 3$	$\omega \odot \varepsilon = \omega/2$
$1/2 \odot 1 = 1/2$	$1/2 \odot 2 = 5/8$	$1/2 \odot 3 = 21/32$	$1/2 \odot \omega = 2/3$
$1/4 \odot 1 = 1/4$	$1/4 \odot 2 = 9/32$	$1/4 \odot 3 = 73/256$	$1/4 \odot \omega = 2/7$

Literatura

[1] Conway, J. H.: *On Numbers and Games.* Academic Press, London, 1976, 1979.

[2] Cihlář J.,–Vopravil V.: *Hry a čísla.* Pedagogická fakulta, Ústí nad Labem, 1983.

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On One Construction of Rational Numbers

Václav Vopravil

The system of natural numbers has an obvious defect in that, given $a, b \in N$, the equation $a+x=b$ may or may not have a solution. Everyone knows that this state of affairs is remedied by adjoining to the natural numbers (then called positive integers) the additional numbers zero and the negative integers to form the set Z of all integers.

The system of integer numbers has an obvious defect in that, given integers $a \neq 0$ and b, the equation $ax = b$ may or may not have a solution. For example $2x = 4$ has the solution $x = 2$ but $2x = 5$ has no solution. This defect is remedied by adjoining to the integers additional numbers (then called fractions) to form the system Q of rational numbers.

Usual algebraic constructions of the field of rational numbers consist of using of so called the Theorem of Semigroup's Imbedding into a Group. There are some known constructions here:

- The improvement of the structure $(N, +)$ to $(Z, +)$ and then the improvement of the operation of the multiplication of integers to a commutative field of rational numbers $(Q, +, \cdot)$.
- The improvement of the structure $(N-\{0\}, \cdot)$ to the group of „positive rational numbers", the complementation of the zero element and „all negative elements".
- The extension of natural numbers to decimal numbers and then the complementation of decimal numbers by infinite periodic decimal numbers and the complementation of this set by „negative decimal numbers".
- And so on.

This contribution deals with one generalization of the Theorem of Semigroup's Imbedding into a Group. We shall improve both binary operations in the structure $(N, +, \cdot)$. We shall assume the reader knows all properties of the structure N.

The structure $(N, +, \cdot)$ is a commutative semiring of natural numbers completed by the zero element 0 and unit element 1 with a cancellation according to the operation + and with a cancellation by non–zero elements according to the operation $\cdot$.

The Construction of the Set Q

1. First let us define the helping set

$$M =_D N \times N \times (N - \{0\}) = \{(x, y, z);$$

$$x, y \in N \wedge z \in N - \{0\}\}.$$

2. Let us define the relation $\sim$ at the set M in this way:

$$(\forall (x, y, z), (a, b, c) \in M) \quad (x, y, z) \sim (a, b, c) \Longleftrightarrow$$

$$\Longleftrightarrow_D xc + bz = az + yc.$$

3. The relation $\sim$ defined at the set M is equivalence.
We shall show by degrees that this relation is reflexive, symmetric and transitive.

Let us select an arbitrary triplet $(x, y, z) \in M$. Then

$$(x, y, z) \sim (x, y, z) \Longleftrightarrow xz + yz = xz + yz.$$

The assertion evidently holds and the relation $\sim$ is reflexive at the set M. □

Let us assume further that for the selected triplets $(x, y, z), (a, b, c) \in M$ holds $(x, y, z) \sim (a, b, c)$. According to the definition of relation $\sim$ holds $xc + bz = az + yc$. We shall prove that the assumption $(a, b, c) \sim (x, y, z)$ is right, i.e. $az + yc = xc + bz$. The last equality holds thanks to the symmetry of relation of equality at the set N. □

The last assertion will be more difficult: We have to prove

$(\forall (x, y, z), (a, b, c), (k, l, m) \in M) \quad (x, y, z) \sim (a, b, c) \wedge (a, b, c) \sim (k, l, m) \Longrightarrow (x, y, z) \sim (k, l, m).$

Let us select arbitrary three triplets $(x, y, z), (a, b, c), (k, l, m) \in M$ so that the precondition of the proving implication was fulfilled. Thanks to the definition of the relation $\sim$ we assume validity of these two assumptions

$$xc + bz = az + yc \tag{1}$$

$$am + lc = kc + bm. \tag{2}$$

We shall prove $(x, y, z) \sim (k, l, m)$, too. According to the definition of the relation $\sim$ we have to show that $xm + lz = kz + ym$.
At first we arrange the equalities (1), (2) so that we multiplicate (1) by m and (2) by z. We shall get

$$xcm + bzm = azm + ycm \tag{3}$$

$$amz + lcz = kcz + bmz. \tag{4}$$

Now we shall add lcz with equality (3) and we shall add ycm with equality (4) and we shall get

$$xcm + bzm + lcz = azm + ycm + lcz \tag{5}$$

$$amz + lcz + ycm = kcz + bmz + ycm. \tag{6}$$

Thanks to the transitivity of the relation $=$ at the set N it holds[1]:

$$xcm + bzm + lcz = kcz + bmz + ycm.$$

If we shall reduce by bmz, we shall get $xcm + lcz = kcz + ycm$, thanks to distributive, associative and commutative etc. properties $c(xm + lz) = c(kz + ym)$ and if we reduce by a non-zero element $c \in N$, we shall get $xm + lz = kz + ym$, i.e. $(x, y, z) \sim (k, l, m)$ according to the definition of relation $\sim$. □

4. The relation $\sim$ creates a decomposition of the set M, i.e.
$M \mid_\sim = \{[x, y, z]\}_{(x,y,z)\in M}, [x, y, z] = \{(a, b, c) \in M;\ (a, b, c) \sim (x, y, z)\}$.

Let us notice that every considering class $[x, y, z]$ is a set and that's why

$$[x, y, z] = [a, b, c] \text{ iff } (x, y, z) \sim (a, b, c).$$

Exercise 1 *Decide, how do the elements of some concrete class look, e.g.*

$$[3, 2, 1] = \{(3, 2, 1), (2, 1, 1), (1, 0, 1), (9, 6, 3), (3, 0, 3), \ldots\}.$$

[1] The reader had found that we did not justify our arrangement in detail now; we had used gradually distributive and asociative etc. properties of the structure N.

The reader will make one's self sure of validity of our conclusions and will try to make the survey on next classes too. The reader will find that

$(x, y, z) \sim (\alpha \cdot (x + \beta), \alpha \cdot (y + \beta), \alpha \cdot z)$, for all $x, y, \beta \in N$, $z, \alpha \in N - \{0\}$.

Let us indicate the decomposition of $M \mid_\sim$ by $\mathcal{Q}$, i.e. $M \mid_\sim = \mathcal{Q}$. Let us betray without overtaking that this set is a support of rational numbers. The impatient reader can think now, why is the relation $\sim$ acceptably guessed.

The Structure $\mathcal{Q}$

Let us define two suitable binary operations at the set $\mathcal{Q}$ so that the new structure $(\mathcal{Q}, \oplus, \otimes)$ would be a commutative field. At first let us pay attention to the operation $\oplus$, which we will shortly name the addition[2]

$$(\forall [x, y, z], [a, b, c] \in \mathcal{Q}) \;\; [x, y, z] \oplus [a, b, c] =_D [xc + az, yc + bz, zc].$$

We'll show the structure $(\mathcal{Q}, \oplus)$ is an Abelian group.

1. The Structure $(\mathcal{Q}, \oplus)$ is commutative.
   Let us select arbitrary classes $[x, y, z], [a, b, c] \in \mathcal{Q}$. Now we arrange

   $$[a, b, c] \oplus [x, y, z] = [az + xc, bz + yc, cz] = [x, y, z] \oplus [a, b, c],$$

   — because thanks to the commutative property of operations $+$ and $\cdot$ at the set N and thanks to the definition $\oplus$ at $\mathcal{Q}$. □

2. The structure $(\mathcal{Q}, \oplus)$ is associative.
   Here we may use commutative, associative and distributive properties of the structure $(N, +, \cdot)$. We shall gradually get $([x, y, z] \oplus [a, b, c]) \oplus [k, l, m] = [xc + az, yc + bz, zc] \oplus [k, l, m] = [(xc + az)m + k(zc), (yc + bz)m + l(zc), (zc)m] = [xcm + azm + kzc, ycm + bzm + lzc, zcm] = [xcm + amz + kcz, ycm + bmz + lcz, zcm] = [x, y, z] \oplus [am + kc, bm + lc, cm] = [x, y, z] \oplus ([a, b, c] \oplus [k, l, m])$.
   We let the justification in detail for the reader again. □

3. The structure $(\mathcal{Q}, \oplus)$ has the zero element $[0, 0, 1]$.
   In fact,

   $$[x, y, z] \oplus [0, 0, 1] = [x \cdot 1 + 0 \cdot z, y \cdot 1 + 0 \cdot z, z \cdot 1] = [x, y, z],$$

   according to the properties of the elements $1, 0 \in N$. □

   Exercise 2 *Try to show* $[x, x, z] = [x, x, z \cdot z]$.

[2]The reader will make one's self sure this formula is the definition of the binary operation, which does not depend on the selection of the representants of classes.

4. The structure $(\mathcal{Q}, \oplus)$ has the property of inverse elements.
   The inverse element of the element $[x, y, z]$ is the class $[y, x, z]$. Thanks to the commutative property of the operation $+$ at the set N it holds true $xz+yz = yz+xz$, whence $(xz+yz)\cdot 1+0\cdot z\cdot z = 0\cdot z\cdot z+(yz+xz)\cdot 1$, i.e.
   $(xz + yz, yz + xz, z \cdot z) \sim (0, 0, 1)$ and whence $[x, y, z] \oplus [y, x, z] = [0, 0, 1]$. □

   Let us accept we will mark the inverse element $[y, x, z]$ of the element $[x, y, z]$ as $\ominus[x, y, z]$, therefore it holds true $\ominus[x, y, z] = [y, x, z]$. □

Conclusion 1 *The structure $(\mathcal{Q}, \oplus)$ is an Abelian group.*

Exercise 3 *Show, that $\{[x, y, 1]\}_{x,y\in N} = \mathcal{Q}_1$ is isomorphic with an additive group of integers Z.*

Now we will define next binary operation $\otimes$ at the set $\mathcal{Q}$, which we will shortly name the multiplication[3].

$$(\forall[x, y, z], [a, b, c] \in \mathcal{Q}) \quad [x, y, z] \otimes [a, b, c] =_D [xa + yb, ya + xb, zc].$$

1. The structure $(\mathcal{Q}, \otimes)$ is commutative.
   We get $[a, b, c] \otimes [x, y, z] = [ax + by, bx + ay, cz] = [x, y, z] \otimes [a, b, c]$ thanks to the commutativity of the operations $+$ and $\cdot$ at N. □

2. The structure $(\mathcal{Q}, \oplus, \otimes)$ is distributive.
   We let to the reader the justification of this assertion.

3. The structure $(\mathcal{Q}, \otimes)$ is associative.
   We let to the reader the justification of this assertion, too.

4. The structure $(\mathcal{Q}, \otimes)$ has the unit element and this element is the class $[1, 0, 1]$.
   It holds namely
   $$[x, y, z] \otimes [1, 0, 1] = [x \cdot 1 + y \cdot 0, y \cdot 1 + x \cdot 0, z \cdot 1] = [x, y, z].$$
   □

5. The structure $(\mathcal{Q} - \{0\}, \otimes)$ has the property of inverse elements
   $$[x, y, z]^{-1} = [zx, zy, (x^2 - 2xy + y^2)],$$
   i.e.
   $$[x, y, z]^{-1} = \begin{cases} [z, 0, x - y] & \text{for } x > y \\ [0, z, y - x] & \text{for } x < y. \end{cases}$$
   We let again the ratification for the reader. It holds for instance

[3] The reader will make one's self sure of that: The binary operation at the set $\mathcal{Q}$ is defined by this formula, too.

$$[x, 0, z]^{-1} = [z, 0, x]$$

or

$$[0, y, z]^{-1} = [0, z, y].$$

Conclusion 2 *The structure* $(\mathcal{Q}, \oplus, \otimes)$ *creates the commutative field.*

The Extension of N

Let us show further, that the structure $\mathcal{Q}$ is the extension of structure the N. Let us limit only on the set $\mathcal{Q}_{01} = \{[x, 0, 1]\}_{x \in N}$. The structure $(\mathcal{Q}_{01}, \oplus, \otimes)$ is isomorphic with the structure $(N, +, \cdot)$. The relation $\varphi \subseteq N \times \mathcal{Q}_{01}$, which is defined in this way:

$$(\forall x \in N) \quad \varphi(x) = [x, 0, 1],$$

is the wanted isomorphic transformation.

We will show the φ is bijective.

Let us select $x, y \in N$, then

$$\varphi(x) = \varphi(y) \iff [x, 0, 1] = [y, 0, 1] \iff (x, 0, 1) \sim (y, 0, 1) \iff$$
$$(x + 0) \cdot 1 = (y + 0) \cdot 1 \iff x = y.$$

Thus an $x \in N$ exists for arbitrary $[x, 0, 1] \in \mathcal{Q}_{01}$ so that $\varphi(x) = [x, 0, 1]$. □

Let us select further $x, y \in N$. Then

$\varphi(x) \oplus \varphi(y) = [x, 0, 1] \oplus [y, 0, 1] = [x \cdot 1 + y \cdot 1, 0 \cdot 1 + 0 \cdot 1, 1 \cdot 1] = [x + y, 0, 1] = \varphi(x + y)$

and

$\varphi(x) \otimes \varphi(y) = [x, 0, 1] \otimes [y, 0, 1] = [xy + 0 \cdot 0, 0 \cdot y + x \cdot 0, 1 \cdot 1] = [xy, 0, 1] = \varphi(x \cdot y).$ □

Because the fact that structures N and $\mathcal{Q}_{01}$ are isomorphic and therefore algebraicly undistinguished we can identify the corresponding elements $x \equiv [x, 0, 1]$ and corresponding operations too. This agreement has got its consequence for all elements of the original structure $\mathcal{Q}$. An arbitrary class $[x, y, z]$ can be written in this way:

$[x, y, z] = ([x, 0, 1] \oplus [0, y, 1]) \otimes [1, 0, z] = ([x, 0, 1] \oplus (\ominus[y, 0, 1])) \otimes [z, 0, 1]^{-1}.$

If we use the agreement concerning the identification of elements and operations, we get

$$[x, y, z] = (x + (-y)) \cdot z^{-1}.$$

To get an ordinary indication, let's define a new binary operation subtraction as an addition of an opposite element and a new binary operation division as a multiplication with an inverse element. Then we have

$$[x, y, z] = \frac{x - y}{z},$$

where $x, y \in N, \quad z \in N - \{0\}$.

Exercise 4 *Prove the structure $\mathcal{Q}$ is isomorphic with the structure Q.*

Exercise 5 *Show, that $\{[x, y, 1]\}_{x,y \in N} = \mathcal{Q}_1$ is isomorphic with the integral domain of integers.*

Exercise 6 *Show, that $\{[x, 0, z]\}_{(x,z) \in (N-\{0\})^2}$ is isomorphic with the multiplicative group Q^+.*

Exercise 7 *Arrange the structure $\mathcal{Q}$ and show that this arrangment is only possible and it is the extension of natural numbers.*

Exercise 8 *Use the ideas of the proof and try to create rational number so that*

$$[x, y, z] = \frac{x}{y - z},$$

respectively

$$[x, y, z] = x - \frac{y}{z}.$$

Use the helping set $M = N \times (N \times N- =)$, respectively $M = N \times N \times (N - \{0\})$.